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FIXED POINTS OF EXPANSIVE ANALYTIC MAPS (II)

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The main result of this note, the "Encircling Theorem," states conditions that assure that an analytic function is expansive in the closed unit disk D and has a fixed point in D. A corollary describes in detail the case of a conformal map. From a new covering lemma for polynomials further sufficient conditions are deduced that guarantee that a polynomial of degree n, n = 1,2,..., is expansive and has a fixed point in D. On the other hand, an important example shows that for each n ≥ 3 polynomials of degree n exist that cover D but do not have a fixed point in D. Finally, the distribution of the fixed points of any finite Blaschke - product is established.

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TABLE OF CONTENTS

Page

LIST OF FIGURES v

1. INTRODUCTION 1

2. FEJÉR'S REMARK 3

3. THE "ENCIRCLING" THEOREM 4

4. A COVERING LEMMA 6

REFERENCES 10

APPENDIX 1 11

APPENDIX 2 13

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LIST OF FIGURES

Figure		Page
1.	p covers D	1
2.	q covers K	2
3.	$z + (z + 1.1i)^3$	5
4.	The location of the zeros of G_n	8

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1. INTRODUCTION.

In BRL-TR-3063, Fixed Points Of Expansive Analytic Maps, Nov. 1989 [7], the notion of an "expansive analytic map" was introduced. Let D be the closed unit disk in the complex plane and f an analytic function in D. By definition, f is "expansive" in D if the image $f(D)$ satisfies the condition $f(D) \supseteq D$. Equivalently, we say that f "covers" D whenever $f(D) \supseteq D$ holds. The principal result in BRL-TR-3063 states that a quadratic complex polynomial that covers D has a fixed point in D. More generally, it was shown that the result holds for any closed disk K, i.e., $f(K) \supseteq K$ implies that there exists $z_0 \in K$ such that $f(z_0) = z_0$. Figure 1. and Figure 2. illustrate the principal result for D and K: $|z-1| \leq 1$. The quadratic polynomials chosen are $p(z) = z + ie^{-i\frac{5}{8}\pi}(z - e^{i\frac{\pi}{4}})(z+1)$ and $q(z) = z^2 - \frac{3}{2}z + \frac{3}{2}$, respectively.

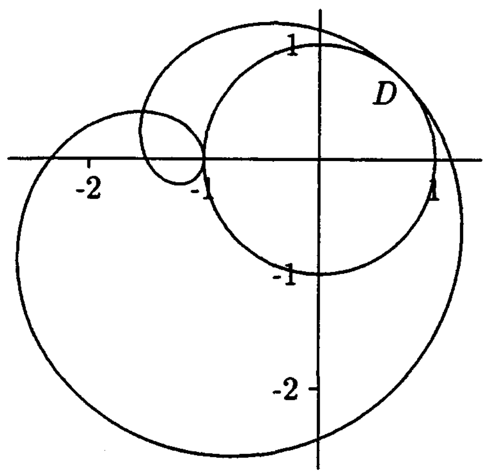

Figure 1. p covers D

The purpose of this note is to give further results on expansive maps. The main result, given in Section 3., is the "Encircling" Theorem. It states that if for an analytic function f in D (not necessarily a polynomial) it is known that it maps an interior point of D into the open unit disk and the image of the unit circle "encircles" the unit circle ("f pushes the unit circle out"), then f is expansive and has a fixed point in D. For example, the function of Figure 1. has this "encircling" property, whereas the function of Figure 2. does not have this property. As a corollary to the "encircling" theorem we

obtain a proof of an important theorem for univalent expansive maps that is independent of Brouwer's fixed point theorem for the plane and more informative than a weaker version obtainable from this theorem. The proof of the "encircling" theorem uses at key junctures a pertinent but not widely publicized remark of Fejér regarding Rouché's theorem. In Section 2. we state Fejér's addendum to Rouché's theorem. It is a consequence of Fejér's remark that a finite Blaschke-Product has a fixed point on the unit circle. We proved that if the number of factors is n, then at least $n-1$ fixed points are on the unit circle. This result was proposed to the problem section of the American Mathematical Monthly and appeared as Problem 6654 on page 273 of the March 1991 issue (Vol. 98, Number 3) of this journal. Our solution, as submitted to the editors, is reproduced in Appendix 1. In Section 4. we prove a covering lemma for polynomials and derive a sufficient condition for a polynomial of degree n to cover D and have a fixed point in D. We also define for each $n \geq 3$ a polynomial g_n that covers D but does not have a fixed point in D. This example shows that the principal result of BRL-TR-3063 cannot be generalized to polynomials of degree $n \geq 3$ without additional assumptions. Finally, in Appendix 2, we give a proof of Lemma F that was left to the reader in BRL-TR-3063. This particular proof keeps the necessary algebraic manipulations to a minimum.

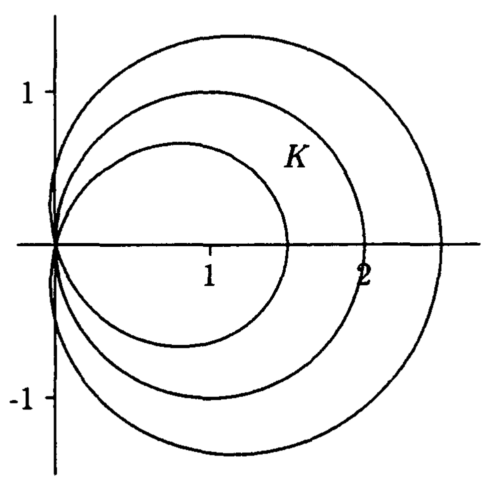

Figure 2. q covers K

2. FEJÉR'S REMARK.

What we call "Fejér's Remark" is explicitly attributed to him by Lipka [1,p.143]. It is also recorded in [2,p.5,Exercise 7]. If we introduce the notation $C = \{ z \mid |z| = 1 \}$ for the unit circle and $U = \{ z \mid |z| < 1 \}$ for the open unit disk, "Fejér's Remark" is the following statement.

Fejér's Remark. Let F and G be analytic in D. If for $z \in C$

$$\begin{array}{rl} (i) & F(z) \neq 0 \\ (ii) & F(z) + G(z) \neq 0 \\ (iii) & |F(z)| \geq |G(z)|, \end{array}$$

then F and $F + G$ have the same number of zeros in U.

For the proof, see [1,p.143].
As a first application, we observe that a "Blaschke-factor"

$$B_n(z) = e^{i\omega} \prod_{k=1}^{n} \frac{z - a_k}{1 - \bar{a}_k z}, \; |a_k| < 1, \; k = 1, 2, \ldots, n, \; n \geq 2$$

has a fixed point on the unit circle $|z| = 1$.
Suppose this were false. Since a Blaschke-factor is expansive as well as non-expansive, the inequality $|B_n(z)| \geq 1 = |-z|$ for $z \in C$ would imply that B_n has at least two fixed points in U, while the inequality $|z| \geq 1 = |-B_n(z)|$ for $z \in C$ would imply that B_n has exactly one fixed point in U. This contradiction proves the assertion. A more detailed analysis shows that B_n has at least $n - 1$ fixed points on the unit circle (see Problem 6654 in the March 1991 issue of the American Mathematical Monthly and Appendix 1. of this note).
As a second application, we remark that for analytic functions the following stronger form of Brouwer's fixed point theorem for the disk holds: If f is analytic in D and $f(D) \subseteq D$, then f has a fixed point in D. If f does not have a fixed point on the unit circle, then f has a unique fixed point in the open unit disk. For the proof, we apply Fejér's remark to the functions $F(z) = -z$ and $G(z) = f(z)$.

3. THE "ENCIRCLING" THEOREM.

Theorem ("encircling" theorem). Let f be analytic in D and $|f(a)| < 1$ for some $|a| < 1$. If $|f(C)| \geq 1$, where $|f(C)| := |f(e^{i\phi})|$, $0 \leq \phi < 2\pi$, then

(*i*) $f(D) \supseteq D$.
(*ii*) f has a fixed point in D.
(*iii*) If f does not have a fixed point on C, f has as many fixed points in U as f has zeros in U.

Proof. Since $|f(C)| \geq 1$, the inequality $|f(z)| \geq 1 > |-f(a)|$ holds for $z \in C$. Therefore, by Rouché's theorem, f has a zero in U. If $u_0 \in U$, it follows from $|f(z)| \geq 1 > |-u_0|$ for $z \in C$, again by Rouché's theorem, that f covers U since u_0 is arbitrary. If $u_0 \in C$, either $f(e^{i\phi_0}) = u_0$ for some ϕ_0 or, by Fejér's remark, $f(z_0) = u_0$ for some $z_0 \in U$. This proves (*i*). Suppose (*ii*) were false, i.e., assume it were possible that $f(z) \neq z$ for $z \in D$. Then, in particular, we would have $f(z) - z \neq 0$ for $z \in C$ and the inequality $|f(z)| \geq 1 = |-z|$ would imply, by Fejér's remark (set $F(z) = f(z)$, $G(z) = -z$), that $f(z)$ and $f(z) - z$ have the same number of zeros in U. Since f has at least one zero in U by (*i*), $f(z) - z = 0$ for some $z_0 \in U$, i.e., f has at least one fixed point in U. This contradiction proves (*ii*) and (*iii*) is immediate. This completes the proof.

Corollary. Let f be analytic and univalent in D. If f is not a conformal automorphism of U and $f(D) \supseteq D$, then

(*i*) f has a fixed point in D.
(*ii*) If f does not have a fixed point on the unit circle C, f has a unique fixed point $z_0 \in U$ such that $|f'(z_0)| > 1$, i.e., z_0 is a repeller.
(*iii*) If $f(a) = 0$, $a \neq 0$, and (*ii*) occurs, the repeller z_0 lies in the "annular sector" A_s defined by

$$A_s = \left\{ z \;\middle|\; \begin{array}{c} |a|\left(1+\sqrt{1-|a|^2}\right)^{-1} < |z| < 1 \\ \cos\left(\arg(z/a)\right) > |a| \end{array} \right\}.$$

Proof. Clearly $|f(C)| \geq 1$. Since f covers D and is univalent, there exists a unique $a \in U$ such that $f(a) = 0$. Therefore, (i) follows by the "encircling" theorem. If f does not have a fixed point on C, it follows from (iii) of the "encircling" theorem that f has a unique fixed point $z_0 \in U$ because it has the unique zero $a \in U$. Since $f(D) \supseteq D$ implies $f^{-1}(D) \subseteq D$, the inverse $g = f^{-1}$ of f is bounded by 1 and $g(z_0) = z_0$. The Schwarz-Pick lemma [4,5] applies to g and g' satisfies the strict inequality

$$(1 - |z|^2)|g'(z)| < 1 - |g(z)|^2 \quad (z \in U)$$

because g is not a conformal automorphism of U. Setting $z = z_0$, $|g'(z_0)| < 1$, and, hence, $|f'(z_0)| > 1$ follows. This proves (ii). To prove (iii), we note that by Schwarz's lemma z_0 is subject to the inequality $|z_0 - a| < |z_0||1 - \bar{a}z_0|$. Hence, $z_0 \in A_s$. This completes the proof of the Corollary.

Remarks.

1. If an analytic function in D covers D and does not have a fixed point in D, then necessarily $f(C) \cap U \neq \emptyset$. This intersection property may be observed for the function of Figure 3., where the closed curve shown is the image of C under f.
2. $p(z) = z + (1+z)^3$ illustrates (i) and $p(-z)$ illustrates (ii) and (iii) of the Corollary.
3. The proof of the Corollary applies verbatim to analytic f (not necessarily univalent) for which $f(D) \subseteq D$ with an attractor and $f(0)$ taking the place of the repeller and a. This strengthens, in particular, the perennial problem 197[6,p.142].

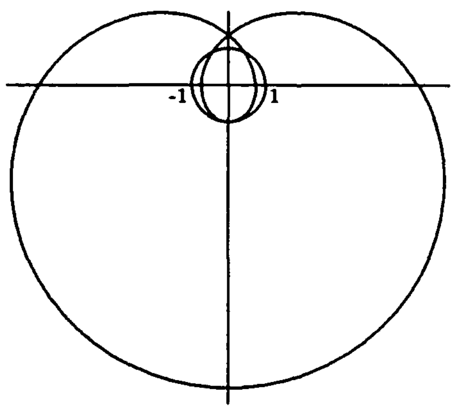

Figure 3. $z + (z+1.1i)^3$

4. A COVERING LEMMA.

The following covering lemma for polynomials is motivated by the elementary observation that every polynomial $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ such that $|a_n| \geq 1 + |a_0|$ covers D and has a fixed point in D.

Lemma. Let $p_n(z) = a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$, $a_n \neq 0$, be a polynomial of degree $n \geq 3$ and such that $|a_1| + |a_{n-1}| > 0$. Then, for some $R \geq R_0 \geq |a_n|$, $p_n(D) \supseteq D_R = \{z \mid |z| \leq R\}$. The radius R_0, depending only on $|a_1|$, $|a_{n-1}|$, and $|a_n|$, is given by

$$R_0 = \begin{cases} \frac{1}{2}\left(|a_{n-1}| + \sqrt{(|a_{n-1}| - 2|a_n|)^2 + 4(|a_{n-1}| - |a_1|)|a_n|}\,\right) & (1) \\ \qquad\qquad if\ |a_{n-1}| > |a_1| \geq 0 & \\ |a_n| \qquad\qquad if\ |a_1| = |a_{n-1}| \leq 2|a_n| & (2) \\ |a_{n-1}| - |a_n| \qquad if\ |a_1| = |a_{n-1}| > 2|a_n| & (3) \\ \frac{1}{2}\left(-|a_{n-1}| + \sqrt{(|a_{n-1}| + 2|a_n|)^2 + 4(|a_1| - |a_{n-1}|)|a_n|}\,\right) & (4) \\ \qquad\qquad if\ |a_1| > |a_{n-1}| \geq 0. & \end{cases}$$

If p_n is such that the inequalities

$$|a_1| > |a_{n-1}| > 2|a_n| \text{ and } (2|a_n| - |a_1|)^2 \geq |a_1|^2 - |a_{n-1}|^2 \qquad (4')$$

both hold, p_n covers an annulus $A(R_1, R_2) = \{\, z \mid R_1 \leq |z| \leq R_2\}$ in addition to the disk D_{R_0}, i.e., $p_n(D) \supseteq D_{R_0}$ and $p_n(D) \supseteq A(R_1, R_2)$. The radii $R_1 > R_0$ and $R_2 \geq R_1$ are given by the formula

$$R_{1,2} = \frac{1}{2}\left(|a_{n-1}| \mp \sqrt{(2|a_n| - |a_1|)^2 + |a_{n-1}|^2 - |a_1|^2}\right).$$

Proof: Using the notation of [3,§6.8,pp.491-496], it follows from the second Schur transform T^2p of the polynomial $p(z) = p_n(z) + a_0$ for some $a_0 \neq 0$ that

$$\gamma_2 = (|a_0|^2 - |a_n|^2)^2 - |\overline{a_0}a_{n-1} - \overline{a_1}a_n|^2.$$

Hence,

$$\gamma_2 \le \gamma_2^* = (|a_0|^2 - |a_n|^2)^2 - (|a_0||a_{n-1}| - |a_1||a_n|)^2.$$

γ_2^* is a polynomial of degree four in $|a_0|$ and satisfies $\gamma_2^* \le 0$ in the interval $|a_n| \le |a_0| \le R_0$. Hence, by Theorem 6.8b [3,p.493], the equation $p_n(z) + a_0 = 0$ has a root in D for every a_0 whose modulus falls into this interval, i.e., p_n covers D_{R_0}. It is clear that there exist polynomials for which the inequalities (4′) both hold. For such polynomials $\gamma_2^* \le 0$ in the disjoint intervals $|a_n| \le |a_0| \le R_0$ and $R_1 \le |a_0| \le R_2$, where $R_1 > R_0$. Therefore, for any a_0 whose modulus lies in one of these intervals, the equation $p_n(z) + a_0 = 0$ has a root in D by Theorem 6.8b, i.e., p_n covers the disk D_{R_0} and the annulus $A(R_1, R_2)$. This completes the proof of the lemma.

Remarks.

1. Similar propositions may be stated for "gap"-polynomials of the form $p_n = a_l z^l + \cdots + a_{n-2}z^{n-2} + a_n z^n$, where $a_n \neq 0$, $a_l \neq 0$, $l > 1$, and $|a_k| \ge 0$, $k = l+1, \ldots, n-2$.
2. For quadratic polynomials $p_2(z) = a_1 z + a_2 z^2$, $a_1 \neq 0$, R_0 is given by $R_0 = |a_2|$ if $|a_1| \le 2|a_2|$ and $R_0 = |a_1| - |a_2|$ if $|a_1| > 2|a_2|$. This result is sharp.
3. $R_0 > |a_n|$ if cases (1), (3) or (4) occur.
4. The lower bound R_0 is independent of the "intervening" coefficients $a_2, a_3, \ldots, a_{n-2}$ and in many cases sharp, e.g., $p_4(z) = z^4 + 3z^3 + 3z^2 + 3z$ covers D_{R_0} with $R_0 = 2$, but there is no $R > 2$ such that $p_4(D) \supseteq D_R$.
5. Every polynomial $p(z) = p_n(z) + a_0$, where p_n satisfies the conditions of the lemma and a_0 is a given constant, covers D for which $R_0 \ge 1 + |a_0|$. Since for the polynomial in Figure 3. this inequality holds, this remark provides an analytical proof of the "geometric evidence" that this polynomial covers D.
6. The lemma may also be invoked to show the existence of fixed points under certain circumstances. For example, let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a given polynomial and set $\widetilde{p_n}(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + (a_1 - 1)z$. Then $p(z) - z = \widetilde{p_n} + a_0$. If $|a_{n-1}| + |a_1 - 1| > 0$, the lemma applies to $\widetilde{p_n}$ and determines an $\widetilde{R_0}$. If $\widetilde{R_0} \ge |a_0|$, p has a fixed point in D.
7. For practical applications we have the following proposition.

Proposition. Let $p, p_n, \widetilde{p_n}$ be defined as in Remarks 5. and 6. Then p covers D and has a fixed point in D if $R_0 \ge 1 + |a_0|$ and $\widetilde{R_0} \ge |a_0|$.

Example: $p(z) = z^n - 4iz^{n-1} + a_{n-2}z^{n-2} + \cdots + a_2z^2 - 3z + (1+\sqrt{2})e^{i\frac{\pi}{4}}$.
For this example $R_0 = 2 + \sqrt{2}$ and $\widetilde{R_0} = 3$. Therefore, p covers D and has a fixed point in D for every $n \geq 3$ and arbitrary $a_2, \cdots, a_{n-2}$.
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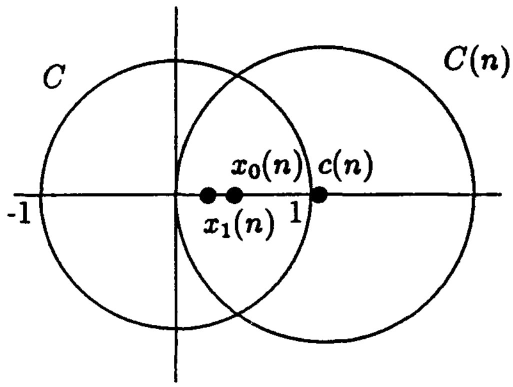

Figure 4. The location of the zeros of G_n.

The following example shows that for each $n \geq 3$ there are polynomials of degree n that cover D but do not have a fixed point in D. We define the polynomials

$$g_n(z) = z + 3(z - e^{i\alpha}(1+n^{-4}))^n$$

$$\alpha = \begin{cases} 0 & if\ n = 2m \quad , \quad m = 2,3,\ldots \\ \pi/2m & if\ n = 2m+1 \quad , \quad m = 1,2,\ldots \end{cases}$$

and prove that they cover D. Clearly, $g_n(z) \neq z$ for $z \in D$.
The polynomials g_3 and g_4 cover by Remark 5.. Since the proof for the odd case is the same as for the even case, we consider only the latter and n will denote an even integer greater than or equal to 6. It follows first by Rouché's theorem that the zeros of the polynomial $G_n(z) = g_n(z) - u$ for every $u \in D$ are contained in the disk $|z - (1+n^{-4})| < 1 + n^{-4}$. The circle $C(n) = \{\, z \mid |z-(1+n^{-4})| = 1 + n^{-4}\}$ with center $c(n) = 1 + n^{-4}$ intersects the unit circle C in two points with x-coordinate $x_0(n) = (2(1+n^{-4}))^{-1}$ (see Figure 4.). The derivative G_n' vanishes on the real axis at the point

$$x_1(n) = 1 + n^{-4} - (3n)^{-1/n-1}.$$

This point $x_1(n)$ lies to the left of $x_0(n)$ for $n \geq 6$. Hence, by the theorem of Gauss-Lucas, there is at least one zero of G_n in the unit disk for every $u \in D$, i.e., g_n covers D. This completes the proof.
This example shows that the principal result of BRL-TR-3063 cannot be generalized to polynomials of degree $n \geq 3$ without additional assumptions. No such additional assumptions are known at present that are less stringent than those in the "encircling" theorem, its corollary, and the consequences of the covering lemma.

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Appendix 1.

In this appendix we give a solution to Problem 6654 that appeared in the American Mathematical Monthly, Vol. 98, Number 3, March 1991, p. 273 as follows:

6654. Proposed by W. O. Egerland and C. E. Hansen, Aberdeen Proving Ground, Aberdeen, MD.

Suppose ω is real, n is a positive integer greater than 1, and $a_1, a_2, \ldots, a_n$ are complex numbers with $|a_k| < 1$ for $k = 1, 2, \ldots, n$. Prove that the equation

$$e^{i\omega}(z - a_1)(z - a_2)\cdots(z - a_n) = z(1 - \bar{a}_1 z)(1 - \bar{a}_2 z)\cdots(1 - \bar{a}_n z)$$

has at least $(n-1)$ roots on the unit circle.

Solution by the Proposers. The roots of the given equation are the fixed points of the "Blaschke-Product"

$$B_n(z) = e^{i\omega}\prod_{k=1}^{n}\frac{z - a_k}{1 - \bar{a}_k z}.$$

We consider the cases 1. $B_n(0) \neq 0$ and 2. $B_n(0) = 0$ separately.

1. If $B_n(0) \neq 0$, B_n has $(n+1)$ fixed points. Since B_n is analytic in $U = \{z \mid |z| < 1\}$ and satisfies $|B_n(z)| < 1$ for $z \in U$, B_n belongs to the class of "analytic functions of bound 1 in the open unit disk". It is well-known that such functions can have at most one fixed point , if any, in U[1,2,3]. Therefore, in particular, B_n can have at most one fixed point $z_0 \in U$. Furthermore, it follows from the lemma of Schwarz-Pick[4] that the multiplicity of z_0 is 1, by which is meant that the multiplicity of z_0 as a zero of the function $B_n(z) - z$ is 1. According to the lemma, the derivative $B_n'(z)$ satisfies the inequality

$$(1 - |z|^2)|B_n'(z)| < 1 - |B_n(z)|^2 \quad (z \in U).$$

Setting $z = z_0$, $|B_n'(z_0)| < 1$, and hence $B_n'(z_0) - 1 \neq 0$ follows, i.e., z_0 is a zero of multiplicity 1 of $B_n(z) - z$. Finally, we observe that with the exception of $z = 0, z = a_k, z = \bar{a}_k^{-1}, k = 1, 2, \cdots, n$, B_n satisfies the identity

$$B_n(z)\overline{B_n(\bar{z}^{-1})} = 1,$$

so that if z_1, $|z_1| \neq 1$, is a fixed point of B_n, the symmetric point $\bar{z}_1^{-1}$ to z_1 with respect to the unit circle is also a fixed point of B_n. A short calculation yields $B_n'(z_1) = \overline{B_n'(\bar{z}_1^{-1})}$, i.e., both fixed points have multiplicity 1. Since B_n can have at most one such pair of symmetric simple fixed points, the remaining $(n-1)$ fixed points of B_n lie necessarily on the unit circle.

2. If $B_n(0) = 0$, B_n has n fixed points. These are $z = 0$ and exactly $(n-1)$ points on the unit circle $|z| = 1$. This is immediate if $a_1 = a_2 = \cdots = a_n = 0$, and if $a_1 = a_2 = \cdots = a_m = 0$, $m < n$, it follows from

$$B_n(z) - z = z(e^{i\omega} z^{m-1} \prod_{k=m+1}^{n} \frac{z - a_k}{1 - \bar{a}_k z} - 1)$$

$$= z(b_{n-1}(z) - 1)$$

since $|b_{n-1}(z)| \neq 1$ if $|z| \neq 1$. This completes the proof.

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4. Heins, M., Complex Function Theory, Academic Press, New York, 1968, Exercise 7.3, pp. 226-227.

Appendix 2.

Lemma F. Let

$$z^2 + az + b = 0$$

be a quadratic equation with roots $z_1 = \rho_1 e^{i\alpha}$ and $z_2 = \rho_2 e^{i\beta}$.
Then

$$\begin{aligned} P &= \left(|b|^2 - 1\right)^2 - |\bar{a}b - a|^2 \\ &= (1-{\rho_1}^2)(1-{\rho_2}^2)\left((|b| - \cos(\alpha-\beta))^2 + \sin^2(\alpha-\beta)\right). \end{aligned}$$

Proof. Since $a = -\rho_1 e^{i\alpha} - \rho_2 e^{i\beta}$ and $b = \rho_1\rho_2 e^{i(\alpha+\beta)}$, we have

$$\bar{a}b - a = (1-{\rho_2}^2)\rho_1 e^{i\alpha} + (1-{\rho_1}^2)\rho_2 e^{i\beta}$$

and

$$|b|^2 - 1 = {\rho_1}^2{\rho_2}^2 - 1 = (1-{\rho_1}^2)(1-{\rho_2}^2) - (1-{\rho_1}^2) - (1-{\rho_2}^2).$$

Hence

$$\begin{aligned} P &= \begin{array}{lll} (1-{\rho_1}^2)^2(1-{\rho_2}^2)^2 & -2(1-{\rho_2}^2)(1-{\rho_1}^2)^2 & -2(1-{\rho_1}^2)(1-{\rho_2}^2)^2 \\ +(1-{\rho_1}^2)^2 & +(1-{\rho_2}^2)^2 & +2(1-{\rho_1}^2)(1-{\rho_2}^2) \end{array} \\ &\quad -{\rho_2}^2(1-{\rho_1}^2)^2 - {\rho_1}^2(1-{\rho_2}^2)^2 - 2\rho_1\rho_2(1-{\rho_1}^2)(1-{\rho_2}^2)\cos(\alpha-\beta) \\ &= (1-{\rho_1}^2)(1-{\rho_2}^2)({\rho_1}^2{\rho_2}^2 - 1 + 2 - 2\rho_1\rho_2\cos(\alpha-\beta)) \\ &= (1-{\rho_1}^2)(1-{\rho_2}^2)\left((|b| - \cos(\alpha-\beta))^2 + \sin^2(\alpha-\beta)\right). \end{aligned}$$

Corollary 1. The roots of the quadratic equation $z^2 + az + b = 0$ have modulus greater than one if and only if the conditions $|b| > 1$ and $P > 0$ are satisfied. They have modulus less than one if and only if the conditions $|b| < 1$ and $P > 0$ are satisfied.

Corollary 2. A quadratic polynomial with both fixed points on the unit circle covers the unit disk if and only if it has the form

$$f(z) = z + i\lambda e^{-i\frac{\alpha+\beta}{2}}(z - e^{i\alpha})(z - e^{i\beta}), \quad \lambda \neq 0 \text{ and real.}$$

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